Math 138 – Practice Final exam (solutions)

Instructions:

- You have 3 hours to complete this exam.
- No external resources are allowed.
- Do not hesitate to ask for clarification on exam questions.

Question 1. (15 pts)

Please provide a **counterexample and/or disproof** to each of the following incorrect claims—be sure to justify why it is a counterexample/disproof.

1. **(5 pts)** If $(X, \leqslant)$ is a poset with $\#X \leqslant 2^n$, then $(X, \leqslant)$ admits an embedding into $(\mathcal{P}(\{1, \ldots, n\}), \subseteq)$.

2. **(5 pts)** For $a, b \in \mathbb{Z}$ and p a prime number, one has that $v_p(a+b) = \max(v_p(a), v_p(b))$.

3. **(5 pts)** Let $f\colon X \to Y$ be a function and consider $A \subseteq Y$. Then, we have the following equality of subsets of Y
$$A = f(f^{-1}(A)).$$

Solution:

1. Consider $X = \{a, b, c, d\}$ with partial order $\leqslant$ uniquely determined by $a < b < c < d$. Then, $\#X \leqslant 2^2$ but we claim that $(X, \leqslant)$ does not admit an order embedding into $(\mathcal{P}(\{1, 2\}), \subseteq)$. Indeed, observe that the longest chain (i.e., sequence of terms strictly less than their successor) in $(\mathcal{P}(\{1, 2\}, \subseteq)$ is of length 3, either
$$\varnothing \subsetneq \{1\} \subsetneq \{1, 2\},$$
or
$$\varnothing \subsetneq \{2\} \subsetneq \{1, 2\}.$$
If $f\colon X \to \mathcal{P}(\{1, 2\})$ is an order embedding, then as $a < b < c < d$ we would have a chain length 4 given by $f(a) \subsetneq f(b) \subsetneq f(c) \subsetneq f(d)$, but this is impossible as we stated.

   (Alternative solution: consider $X = \{a, b\}$ where a and b are incomparable by $\leqslant$, i.e., neither a nor b are related to each other by $\leqslant$. Then as $\#X = 2 = 2^1$ it would suffice to show that X does not admit an order embedding into $\mathcal{P}(\{1\}), \subseteq)$. But, $\mathcal{P}(\{1\})$ has only two elements $\varnothing$ and $\{1\}$ and those *are comparable*. If $f\colon X \to \mathcal{P}(\{1\})$ were an order embedding then, without loss of generality, $f(a) = \varnothing$ and $f(b) = \{1\}$. But, as $\varnothing \subseteq \{1\}$ as f is an order embedding this would imply $a \leqslant b$, but this is false.)

2. Consider $a = b = p = 2$. Then, $v_2(2) = v_2(2) = 1$, so $\max(v_2(2), v_2(2)) = 1$ but $v_2(2 + 2) = v_2(4) = 2$.

   (Comment: this is just wildly false, the correct statement is actually $v_p(a + b) \geqslant \min(v_p(a), v_p(b))$ and if $v_p(a) \neq v_p(b)$ then, in fact, $v_p(a + b) = \min(v_p(a), v_p(b))$.)

3. Consider $f\colon X \to Y$ where $X = \{a\}$ and $Y = \{1, 2\}$ where $f(a) = 1$. Then, if $A = \{2\}$ we have that $f^{-1}(A) = \varnothing$, and so $f(f^{-1}(A)) = \varnothing$ which is not equal to A.

   (Comment: this condition holds for all A precisely when f is surjective. Good exercise!)

Rubric: For each 5 point question the breakdown is

- **(3 pts)** Correct counterexample.
- **(2 pts)** Correct explanation.

Question 2. (10 pts)

Prove by induction that 6 divides $n^3 - n$ for all $n \geqslant 1$.

Solution: We proceed by induction.

Base case: When $n = 1$ we see that $n^3 - n = 1^3 - 1 = 0$ which is divisible by 6.

Inductive step: Suppose that 6 divides $n^3 - n$. We then aim to show that 6 divides the quantity $(n + 1)^3 - (n + 1)$.

But, observe that

$$(n + 1)^3 - (n + 1) = n^3 + 3n^2 + 2n = (n^3 - n) + (3n^2 + 3n) = (n^3 - n) + 3n(n + 1).$$

Now, by the inductive hypothesis we have that 6 divides $n^3 - n$. Thus, by the above equation we see that 6 divides $(n + 1)^3 - (n + 1)$ if and only if 6 divides $3n(n + 1)$. But, as either n or $n + 1$ is even, we see that 2 divides $n(n + 1)$ and so $2 \cdot 3 = 6$ divides $3n(n + 1)$ as desired.

Rubric:

- **(2 pts)** Correct base case.
- **(4 pts)** Correct induction step idea.
- **(4 pts)** Coherence of explanation.

Question 3. (15 pts)

Let n and m be coprime elements of $\mathbb{N}$ (i.e., they have no common prime divisors). Show that if $x \in \mathbb{N}$ and $\sqrt[m]{x^n}$ is rational, then $\sqrt[m]{x}$ is already rational.

Solution: By what we discussed in class, we know that $\sqrt[m]{x}$ is rational if and only if

- $\operatorname{sgn}(x) = 1$ or $\operatorname{sgn}(x) = -1$ and m is odd,
- for all primes p, we have that m divides $v_p(x)$.

As $x \in \mathbb{N}$ we have that $\operatorname{sgn}(x) = 1$, and so it suffices to show that for all primes p, m divides $v_p(x)$.

Now, by the same result, as $\sqrt[m]{x^n}$ is rational, we have that for each prime p that m divides $v_p(x^n)$. But, $v_p(x^n) = n \cdot v_p(x)$ as $v_p(xy) = v_p(x) + v_p(y)$ as shown in class. Thus, we have that $m \mid n \cdot v_p(x)$. As m and n are coprime, this implies that m divides $v_p(x)$ as desired.

(Alternative solution: As m and n are coprime, we know by Bezout's lemma that there are integers a and b such that $am + bn = 1$. So then, we see that

$$\begin{aligned}
\sqrt[m]{x} &= x^{\frac{1}{m}} \\
&= (x^1)^{\frac{1}{m}} \\
&= (x^{am+bn})^{\frac{1}{m}} \\
&= x^a \cdot ((x^n)^{\frac{1}{m}})^b \\
&= x^a \cdot \sqrt[m]{x^n}^{\,b}.
\end{aligned}$$

Now, by assumption $\sqrt[m]{x^n}$ is rational, x is rational, and a and b are integers. Thus, $\sqrt[m]{x} = x^a \cdot \sqrt[m]{x^n}^{\,b}$ is rational as desired.)

Rubric:

- **(5 pts)** Writing coherence.
- **(5 pts)** Correct idea.
- **(5 pts)** Correctly executed idea.

Question 4. (15 pts)

Let $(X, \leqslant)$ be a poset. Define the *opposite poset* $(X, \leqslant^{\text{op}})$ to have the same underlying set X, but with relation defined by

$$x \leqslant^{\text{op}} y \iff y \leqslant x.$$

1. **(7 pts)** Show that $(X, \leqslant^{\text{op}})$ is a poset.
2. **(8 pts)** Show that if $(Y, \preceq)$ is another poset, then $(X, \leqslant) \simeq (Y, \preceq)$ if and only if $(X, \leqslant^{\text{op}}) \simeq (Y, \preceq^{\text{op}})$. (Recall: $\simeq$ means isomorphism)

Solution:

1. Let us first note that $\leqslant^{\text{op}}$ is reflexive, as $x \leqslant y = x$ and so $x = y \leqslant^{\text{op}} x$.

   To see that $\leqslant^{\text{op}}$ is transitive, suppose that $x \leqslant^{\text{op}} y$ and $y \leqslant^{\text{op}} z$. Then, by definition this means that $x \geqslant y$ and $y \geqslant z$. Thus, by the transitivity of $\leqslant$ we see that $x \geqslant z$ and so, again by definition, $x \leqslant^{\text{op}} z$ as desired.

   Finally, to show that $x \leqslant^{\text{op}}$ is anti-symmetric, assume that $x \leqslant^{\text{op}} y$ and $y \leqslant^{\text{op}} x$. Then, by definition, $x \geqslant y$ and $y \geqslant x$, and so by the anti-symmetry of $\leqslant$ we deduce that $x = y$ as desired.

2. Assume that $f \colon X \to Y$ is a bijection. We will show that this is an isomorphism $(X, \leqslant) \to (Y, \preceq)$ if and only if it is an isomorphism $(X, \leqslant^{\text{op}}) \to (Y, \preceq^{\text{op}})$. This clearly implies the claim.

   But, observe that f is an isomorphism $(X, \leqslant) \to (Y, \preceq)$ if and only if we have the following: for $x, y \in X$ we have that $f(x) \preceq f(y)$ if and only if $x \leqslant y$. As x and y are dummy variables, this is clearly equivalent to $f(y) \preceq f(x)$ if and only if $y \leqslant x$. In turn, this is equivalent to $f(x) \preceq^{\text{op}} f(y)$ if and only if $x \leqslant^{\text{op}} y$. But, this by definition is equivalent to f being an isomorphism $(X, \leqslant^{\text{op}}) \to (Y, \preceq^{\text{op}})$ as desired.

Rubric:

- **(4 pts)** Writing coherence (2 points for each part).
- **(5 pts)** Part 1 correct.
- **(6 pts)** Part 2 correct.

Question 5. (20 pts)

Let $f\colon X \to Y$ be a function. Show that the following are equivalent:

1. f is bijective,
2. for all subsets $A \subseteq X$ the following equality of subsets of Y holds:

$$f(X - A) = Y - f(A).$$

Solution: Let us first assume that f is a bijection. To show that $f(X - A) = Y - f(A)$, let us first assume that $y \in f(X - A)$. By definition this means that there exists $x \in X - A$ such that $y = f(x)$. Clearly then $y = f(x) \in Y$, and to see it's not in $f(A)$ assume otherwise. Then, there exists $a \in A$ such that $y = f(a)$. But, note that $x \neq a$ as $x \notin A$, and thus we have contradicted that f is an injection. Thus, $y \in Y$ and $y \notin f(A)$ so $y \in Y - f(A)$. Conversely, suppose that $y \in Y - f(A)$. As f is surjective there exists $x \in X$ such that $y = f(x)$. Note that $x \notin A$ else $y = f(x) \in f(A)$ which is not true. Thus, $x \in X - A$ and so $y = f(x) \in f(X-A)$. As we have shown both directions, we deduce that $f(X-A) = Y-f(A)$ as desired.

Let us now assume that for all $A \subseteq X$ we have the equality $f(X - A) = Y - f(A)$. To show that f is surjective, take $A = \varnothing$ so that this equality becomes $f(X) = Y$, which is equivalent to surjectivity. To show injectivity, assume that $x_1 \neq x_2$ are in X. Take $A = \{x_1\}$. Then, $x_2 \in X - A$ and so $f(x_2) \in f(X - A)$. But, by assumption this is equivalent to $f(x_2) \in Y - f(A)$ or, that $f(x_2) \notin f(A) = \{f(x_1)\}$. In other words, $f(x_2) \neq f(x_1)$ as desired.

Rubric:

- **(8 pts)** Coherence of explanation.
- **(6 pts)** Correct idea for 1. $\implies$ 2.
- **(6 pts)** Correct idea for 2. $\implies$ 1.

Question 6. (25 pts)

Let $\mathbb{Q}$ be the set of rational numbers. Consider the set $\mathcal{I}$ of all closed intervals with rational endpoints:

$$\mathcal{I} = \{\, [a,b] : a, b \in \mathbb{Q},\ a \leq b \,\}.$$

In the following, you are free to use any facts we proved in class (although state clearly those that you are using).

1. **(10 pts)** Prove that $\mathcal{I}$ is countable.
2. **(10 pts)** Let $\mathcal{U}$ be the set of all finite unions of such intervals, i.e.,

   $$\mathcal{U} = \{\, [a_1, b_1] \cup \cdots \cup [a_n, b_n] \ : \ n \geq 1,\ [a_i, b_i] \in \mathcal{I} \text{ for each } i \,\}.$$

   Prove that $\mathcal{U}$ is countable.
3. **(5 pts)** Briefly explain why this does *not* contradict the fact that there are uncountably many subsets of $[0,1]$.

Solution:

1. Let $S = \{(a,b) \in \mathbb{Q}^2 : a \leqslant b\}$. Consider the function $f\colon S \to \mathcal{I}$ given by $f(a,b) = [a,b]$. By assumption f is a surjection, and thus $\#\mathcal{I} \leqslant \#S \leqslant \#\mathbb{Q}^2 = \aleph_0$, and thus $\mathcal{I}$ is countable.
2. Note that we can write $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ where

   $$\mathcal{U}_n = \{[a_1, b_1] \cup \cdots \cup [a_n, b_n] : [a_i, b_i] \in \mathcal{I} \text{ for all i}\},$$

   i.e., $\mathcal{U}$ is the set of all finite unions of elements in $\mathcal{I}$ and $\mathcal{U}_n$ is the union of exactly n such elements. Note that for each n we have a surjection $f\colon S^n \to \mathcal{U}_n$ given by

   $$f((a_1, b_1), \cdots, (a_n, b_n)) = [a_1, b_1] \cup \cdots \cup [a_n, b_n].$$

   As we proved in class, $\#(\mathbb{Q}^2)^n = \aleph_0$ and so $\#\mathcal{U}_n \leqslant \#S^n \leqslant \aleph_0$ and thus $\mathcal{U}_n$ is countable. As $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a countable union of countable sets we know from class that it's countable.
3. This doesn't contradict that the set of subsets of $[0,1]$ is uncountable because there are many subsets of $[0,1]$ which are not contained in $\mathcal{U}$. For example, consider $\{\frac{1}{\sqrt{2}}\}$, this is not contained in $\mathcal{U}$. Indeed, assume it was. Then, we could write

   $$\{\tfrac{1}{\sqrt{2}}\} = [a_1, b_1] \cup \cdots \cup [a_n, b_n],$$

   for some $n \geqslant 1$ and some $a_1, b_1, \ldots, a_n, b_n \in \mathbb{Q}$. As $[a_i, b_i]$ contains infinitely many elements unless $a_i = b_i$, we see that we must have that $a_i = b_i$ for $i = 1, \ldots, n$. So then,

   $$\{\tfrac{1}{\sqrt{2}}\} = [a_1, b_1] \cup \cdots \cup [a_n, b_n] = \{a_1, \ldots, a_n\}.$$

   But then, observe that the very left-hand side of the above equality contains an irrational number, but the very right-hand side only contains rational numbers, which is a contradiction.

Rubric:

- **(6pts)** Writing coherence (2 pts for each part).
- **(8 pts)** Correct idea for showing countability in 1.
- For 2.:
  - **(5 pts)** Correct idea (for example, trying to write $\mathcal{U}$ as a countable union of countable sets).
  - **(3 pts)** Correctly executing the idea.
- **(3 pts)** For 3., observing that not all subsets are in $\mathcal{U}$ (with an example of something that is not in $\mathcal{U}$).